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VERTEX-VECTORS OF QUADRANGULAR 3-POLYTOPES WITH TWO TYPES OF EDGES

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1. Introduction

If the edge e of the 3-polytope M is incident with vertices A, B and faces α, β, the *type* of e is defined as the ordered couple of unordered number couples $((a, b)(m, n))$ where a, b are valencies of A, B and m, n are numbers of edges of α, β. In the present paper we deal with 3-polytopes having quadrangular faces only and exactly two types of edges, therefore the notation can be simplified. Notice that the vertices of such 3-polytopes M can have at most three different valencies because of the connectedness of the graph of M. Let us therefore denote by $\mathscr{S}(a, b, c)$ the family of all quadrangular 3-polytopes whose all edges are either of the type $((a, b)(4, 4))$ or of the type $((b, c)(4, 4))$.

In [3] the first step in the study of the combinatorial structure of such polytopes has been made. The following result to be used in the sequel has been proved in [3]: *The families* $\mathscr{S}(3, 3, 4)$, $\mathscr{S}(3, 3, 5)$, $\mathscr{S}(3, 3, c)$ *for* $c \geq 11$ *and* $\mathscr{S}(4, 3, 5)$ *are finite. The families* $\mathscr{S}(3, 3, c)$ *for* $6 \leq c \leq 10$, $\mathscr{S}(3, 4, c)$ *and* $\mathscr{S}(3, 5, c)$ *for* $c \geq 4$, $\mathscr{S}(4, 3, c)$ *and* $\mathscr{S}(5, 3, c)$ *for* $c \geq 6$ *are infinite. Every quadrangular* 3*-polytope with exactly two types of edges belongs to precisely one of the families mentioned.*

In the present paper we continue our investigations of the combinatorial structure of quadrangular 3-polytopes with two types of edges and make an attempt to characterize vertex-vectors of such polytopes. (If $v_i(\mathrm{M})$ denotes the number of i-valent vertices of M, $(v_i(\mathrm{M}))$ is the *vertex-vector* of M. In the sequel the superfluous zeros will be left out.) The following sections contain conditions for a triple (v_a, v_b, v_c) or couple (v_a, v_c) of positive integers to be the vertex-vector of a 3-polytope belonging to $\mathscr{S}(a, b, c)$ or $\mathscr{S}(a, a, c)$, respectively. (Notice that $\mathscr{S}(a, b, c) = \mathscr{S}(c, b, a)$.)

Unfortunately, we are unable to present, for certain triples (a, b, c), a complete characterization of vertex-vectors of polytopes belonging to

$\mathscr{S}(a, b, c)$. Therefore we state explicitly all undecided cases. Certain procedures for the construction of planar 3-connected graphs (i.e. of 3-polytopes – see Steinitz's theorem in Grünbaum [2]) are employed. Some general prerequisites must be stated first. To shorten the exposition one more symbol is introduced: Let $G_b(a, c)$ denote the family of all planar maps with a b-regular 2-vertex-connected 3-edge-connected graph whose all faces are either a-gons or c-gons.

Almost all constructions in the sequel use the notion of the *radial map* $r(\mathrm{M})$ of a given planar map M (see e.g. Jucovič [4], Ore [6]). Given a planar map M we associate with M (with the vertex-set $V(\mathrm{M})$, edge-set $E(\mathrm{M})$ and face-set $F(\mathrm{M})$) a map $r(\mathrm{M})$ so that $V(r(\mathrm{M})) = V(\mathrm{M}) \cup F(\mathrm{M})$, and $e = XY \in E(r(\mathrm{M})) \Leftrightarrow X \in V(\mathrm{M})$, $Y \in F(\mathrm{M})$ and X is a vertex of the face Y, or $X \in F(\mathrm{M})$, $Y \in V(\mathrm{M})$ and Y is a vertex of the face X. As every edge $g \in E(\mathrm{M})$ is incident with two vertices and with two faces of M, g determines a quadrangular face of $r(\mathrm{M})$. So for every map M, $r(\mathrm{M})$ is a quadrangular map whose vertex set $V(r(\mathrm{M}))$ is partitioned into two disjoint sets. The valencies of vertices in one set are those of the vertices of $V(\mathrm{M})$, the valencies of the second one are equal to the numbers of edges of the faces from $F(\mathrm{M})$. It is not difficult to prove that if the graph of M is 2-vertex-connected and 3-edge-connected, the graph of $r(\mathrm{M})$ is 3-vertex-connected and therefore realizable as the graph of a 3-polytope (by Steinitz's theorem, see [2]).

We shall use the following lemma which is not hard to deduce from basic relations between M and $r(\mathrm{M})$.

LEMMA 1.1. (a) *If* $\mathrm{M} \in G_b(a, c)$ *then* $r(\mathrm{M}) \in \mathscr{S}(a, b, c)$.

(b) If $\mathrm{P} \in \mathscr{S}(a, b, c)$, $a \neq b \neq c \neq a$, *then there exists a map* $\mathrm{M} \in G_b(a, c)$ *such that* $r(\mathrm{M}) = \mathrm{P}$.

The next lemma (due to Gallai [1]) is employed mainly for proving the nonexistence of a planar map whose radial map belongs to an $\mathscr{S}(a, b, c)$.

LEMMA 1.2. *If all faces of a planar map* M *are p-gons and all vertices of* M *have valencies* $\equiv 0 \pmod q$ *then the number of faces of* M *is an integer multiple of the number of faces of* $\mathrm{P}(p, q)$, *the regular spherical mosaic with all vertices q-valent and all faces p-gons.*

The following lemma is straightforward:

LEMMA 1.3. *Writing* $v_x(\mathrm{M}) = v_x$, *if* $a \neq b \neq c \neq a$ *then for every 3-polytope* $\mathrm{M} \in \mathscr{S}(a, b, c)$

$$av_a + cv_c = bv_b. \tag{1}$$

Manipulations with (1) and with Euler's formula yield necessary conditions contained in

LEMMA 1.4. *The vertex-vector* (v_a, v_b, v_c) *of a polytope* $M \in \mathscr{S}(a, b, c)$, $a \neq b \neq c \neq a$, *satisfies the conditions*

$$v_a = \frac{4b-(2b+2c-bc)v_c}{2a+2b-ab}, \tag{2}$$

$$v_b = \frac{4a+2(c-a)v_c}{2a+2b-ab}. \tag{3}$$

The vertex-vector (v_3, v_c) *of* $M \in \mathscr{S}(3, 3, c)$ *or* $M \in \mathscr{S}(3, c, c)$ *satisfies the condition*

$$v_3 = 8+(c-4)v_c. \tag{4}$$

From Lemma 1.4 it follows that if we look for vertex-vectors (v_a, v_b, v_c) or (v_a, v_c) of all polytopes belonging to $\mathscr{S}(a, b, c)$ or to $\mathscr{S}(a, a, c)$, respectively, we can only examine, for every positive integer m, whether there exists an $M \in \mathscr{S}(a, b, c)$ or $\mathscr{S}(a, a, c)$ such that $v_c(M) = m$. (If so, such an m is called a *suitable value* of v_c.) In the next sections, for every triple (a, b, c) we state the known suitable and unsuitable values of v_c. The reader should try to answer the undecided cases. In the sequel, every triple of integers (v_a, v_b, v_c) which is a candidate for the vertex-vector of an $M \in \mathscr{S}(a, b, c)$ is supposed to satisfy (3) and (2) (and analogously, (4) holds for the pair (v_3, v_c)).

2. The families $\mathscr{S}(4, 3, c)$

Table 1 presents our knowledge of vertex-vectors of quadrangular 3-polytopes belonging to $\mathscr{S}(4, 3, c)$ (all letters denote nonnegative integers). Because of Lemma 1.4, Table 1 deals with the coordinate v_c of these vectors only. (The same applies to other families $\mathscr{S}(a, b, c)$ in Tables 2–4.)

Table 1

The vertex-vectors (v_3, v_4, v_c) of polytopes from $\mathscr{S}(4, 3, c)$

	c	Suitable v_c	Unsuitable v_c	Undecided v_c
1	5	2, 4, 6, 8	all $\neq 2, 4, 6, 8$	–
2	6	all ≥ 2	1	–
3	$8k+i,\ k \geq 1,\ i = 0, 1, 3, 5$	all even ≥ 2	all odd	–
4	$8k+7,\ k \geq 0$	all even ≥ 2	all odd	–
5	$8k+2,\ k \geq 1$	all even ≥ 2 all odd $\geq 4k+3$	1	odd v_c $1 < v_c < 4k+3$
6	$8k+4,\ k \geq 1$	all even ≥ 2 all odd $\geq 2k+5$	1	odd v_c $1 < v_c \leq 2k+3$
7	$8k+6,\ k \geq 1$	all even ≥ 2 all odd $\geq 4k+5$	1	odd v_c $1 < v_c < 4k+5$

Proof of the statements in Table 1.

2.1. First let us deal with the unsuitable values of v_c (column 3).

Certainly $v_c \neq 1$ because there exists no trivalent planar map M with $s_c(\mathrm{M}) = 1$ and $s_i(\mathrm{M}) = 0$ for $i \neq 4$, c, as can be seen by direct construction. ($s_i(\mathrm{M})$ is the number of i-gons of the map M.)

Lines 3 and 4: For the trivalent map M such that $r(\mathrm{M}) = \mathrm{P} \in \mathscr{S}(4, 3, c)$, from Euler's formula it follows that $2s_4(\mathrm{M}) = 12 + (c-6)s_c(\mathrm{M})$; if c is odd, the right-hand side of this equality is even only if $s_c(\mathrm{M}) = v_c(\mathrm{P})$ is even.

If $c \equiv 0 \pmod 8$, the evenness of v_c follows from Lemma 1.2 applied to the dual of the trivalent map from $G_3(4, c)$ whose radial map belongs to $\mathscr{S}(4, 3, c)$.

2.2. All statements in lines 1 and 2 follow from the validity of the appropriate statements on those maps from $G_3(4, c)$ (for maps with 3-connected graphs see Grünbaum [2], Jucovič [4]) whose radial maps belong to $\mathscr{S}(4, 3, c)$.

2.3. We now turn to the statements in the first column (and lines $\neq 1, 2$). In all cases, first a map $\mathrm{M} \in G_3(4, c)$ having v_c c-gons (and quadrangles) is constructed; $r(\mathrm{M})$ is then the desired polytope $\mathrm{P} \in \mathscr{S}(4, 3, c)$.

Procedure 1 increases the number of c-gons in a given map $\mathrm{M} \in G_3(4, c)$ by two. Suppose we have a triple of quadrangles in M as in Fig. 2.1 (a). Insert into the "middle" one $c-4$ new edges as in Fig. 2.1 (b). Two new c-gons appear. This creation of c-gons in pairs can be repeated any number of times.

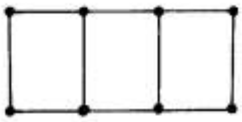

Fig. 2.1(a)

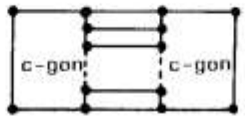

Fig. 2.1(b)

Now, if v_c is even we begin with the c-prism and apply Procedure 1 $(v_c - 2)/2$ times.

To obtain odd numbers of c-gons the starting maps have to be changed.

For $c \equiv 2 \pmod 8$ and $c \equiv 6 \pmod 8$ the starting map (Fig. 2.2) contains $c/2+2$ faces which are c-gons. Further, Procedure 1 is employed.

For $c \equiv 8k+4$ we start with the 6-prism; denote its side-faces by $\alpha_1, \ldots, \alpha_6$ and the bases by β_1, β_2. Decompose each of the faces $\alpha_1, \alpha_3, \alpha_5$ into $4k+1$ quadrangles: $\alpha_{1,1}, \ldots, \alpha_{1,4k+1}, \alpha_{3,1}, \ldots, \alpha_{3,4k+1}, \alpha_{5,1}, \ldots, \alpha_{5,4k+1}$ — the faces $\alpha_2, \alpha_4, \alpha_6$ become $(8k+4)$-gonal. The faces β_1, β_2 are changed into $(8k+4)$-gons as follows: The quadrangles $\alpha_{1,i}$, $i \equiv 1 \pmod 4$, $i \leqslant 4k+1$, are divided by $8k-2$ new edges into $8k-1$ quadrangles; each of the quadrangles $\alpha_{1,i}$, $i \equiv 3 \pmod 4$, is divided by two new edges into three quadrangles. All new edges inserted are to be parallel with those edges of $\alpha_{1,i}$, $i \equiv 1 \pmod 2$, which are common to

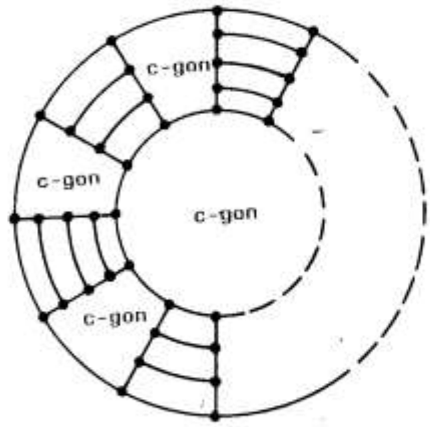

Fig. 2.2

these quadrangles and the face α_2. The above procedure yields $2k+5$ faces which are all $(8k+4)$-gons. Procedure 1 is used to increase the number of c-gons in pairs.

3. The families $\mathscr{S}(5, 3, c)$

The vertex-vectors (v_3, v_5, v_c) of polytopes from $\mathscr{S}(5, 3, c)$

	c	Suitable v_c	Unsuitable v_c	Undecided v_c
1	$10k,\ k \geq 1$	all even	all odd	–
2	$10k+1,\ k = 1, 2, 3$	all even	1	all odd > 1
3	$10k+1,\ k \geq 4$	all $\neq 1, 3, 5, 7, 9, 11, 13, 15, 17, 19$	1	3, 5, 7, 9, 11, 13, 15, 17, 19
4	$10k+j,\ j = 2, 3,\ k = 1, 2$	all even	1	all odd > 1
5	$10k+j,\ j = 2, 3,\ k \geq 3$	all $\neq 1, 3, 5, 7, 9, 11, 13$	1	3, 5, 7, 9, 11, 13
6	$10k+i,\ i = 4, 5,\ k \geq 1$	all ≥ 2	1	–
7	$10k+6,\ k \geq 0$	all ≥ 2	1	–
8	7	all even ≥ 2	1	all odd > 1
9	$10k+7,\ k \geq 1$	all $\neq 1, 3, 5, 7$	1	3, 5, 7
10	8	all ≥ 2	1	–
11	$10k+8,\ k \geq 1$	all $\neq 1, 3, 5, 7$	1	3, 5, 7
12	$10k+9,\ k = 0, 1, 2, 3, 4$	all even ≥ 2	1	all odd > 1
13	$10k+9,\ k \geq 5$	all $\neq 1, 3, 5, 7, 9, 11, 13, 15, 17$	1	3, 5, 7, 9, 11, 13, 15, 17, 19

Proof of the statements in Table 2.

3.1. The nonexistence of $M \in \mathscr{S}(5, 3, c)$ with $v_c(M) = 1$ is demonstrated exactly as the analogous statement for polytopes from $\mathscr{S}(4, 3, c)$.

If $c \equiv 0 \pmod{10}$, the evenness of v_c follows from Lemma 1.2.

3.2. Instead of polytopes from the family $\mathscr{S}(5, 3, c)$ we will again first construct suitable maps M from $G_3(5, c)$, $c \geq 6$. As noted in Lemma 1.1, $r(\mathrm{M}) = \mathrm{P} \in \mathscr{S}(5, 3, c)$ for every $\mathrm{M} \in G_3(5, c)$. It is easy to see that $v_c(\mathrm{P}) = s_c(\mathrm{M})$.

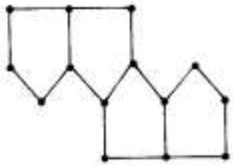

Fig. 3.1(a)

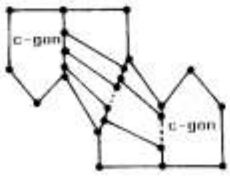

Fig. 3.1(b)

Procedure 2 increases the number of c-gons in a given map $\mathrm{M} \in G_3(5, c)$ by two. Suppose we have in M a quadruple of pentagons as in Fig. 3.1 (a). Add to it $2c-10$ new edges as in Fig. 3.1 (b). Two new c-gons appear. Quadruples of pentagons to be used for repeating the construction appear as well.

If v_c is even, the starting map for every c is the map of the regular dodecahedron. Procedure 2 is performed $v_c/2$ times. For odd v_c the situation is a little more complicated.

3.3. For $v_c \equiv 1 \pmod 2$ we again construct only starting maps from $G_3(5, c)$ with an odd number of c-gons. Each of these maps will contain

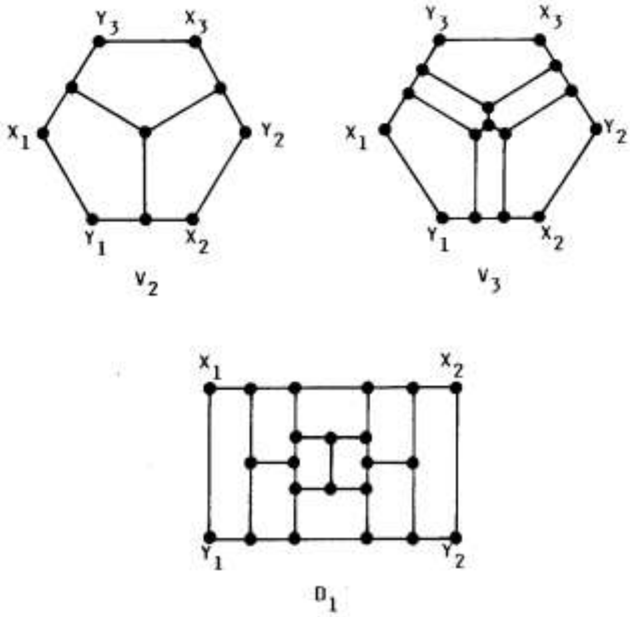

Fig. 3.2

a quadruple of pentagons as in Fig. 3.1 (a), to be able to perform Procedure 2. A construction similar to that of Owens [6] will be used. A basic role is played by the configurations V_2, V_3, D_1 and D_m. The first three of them are shown in Fig. 3.2. The configuration D_m, $m \geqslant 2$, is obtained from D_{m-1} and D_1 by identifying the edge $X_2 Y_2$ of D_{m-1} with the edge $X_1 Y_1$ of D_1 and then deleting these labels. All vertices of these configurations are 3-valent, apart from pairs of adjacent 2-valent vertices $X_i Y_i$, $i = 1, 2, 3$. The edges $X_i Y_i$ which join them will be called *half edges*. All interior faces of these configurations are pentagons.

To construct the required maps from the family $G_3(5, c)$, we take copies of V_m and D_n (with suitable values of m and n) and connect them by identifying half edges. To specify the pattern of joins and the values of m and n, we use a 2-connected 3-valent planar multigraph with suitable labels. A vertex with label m denotes V_m, an edge with label n denotes D_n and incidence between the vertex and the edge indicates that V_m and D_n have a half edge identified. An unlabeled edge (or an edge with label 0) joining vertices with labels m and m'

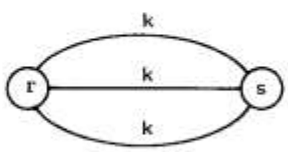

Fig. 3.3

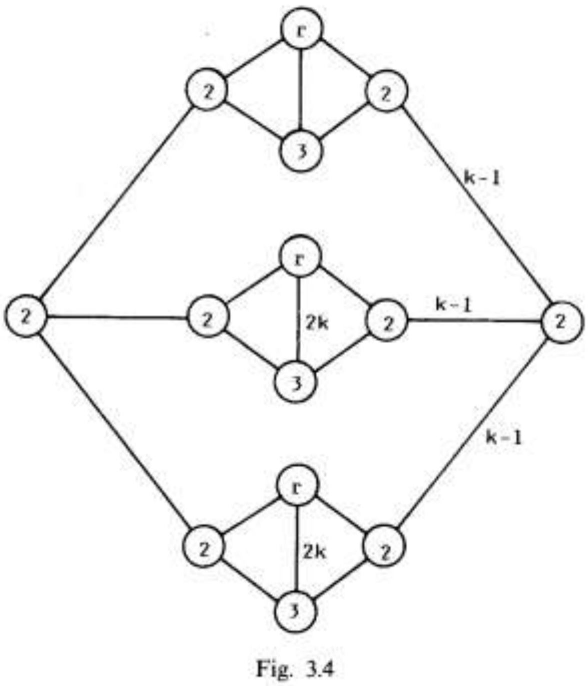

Fig. 3.4

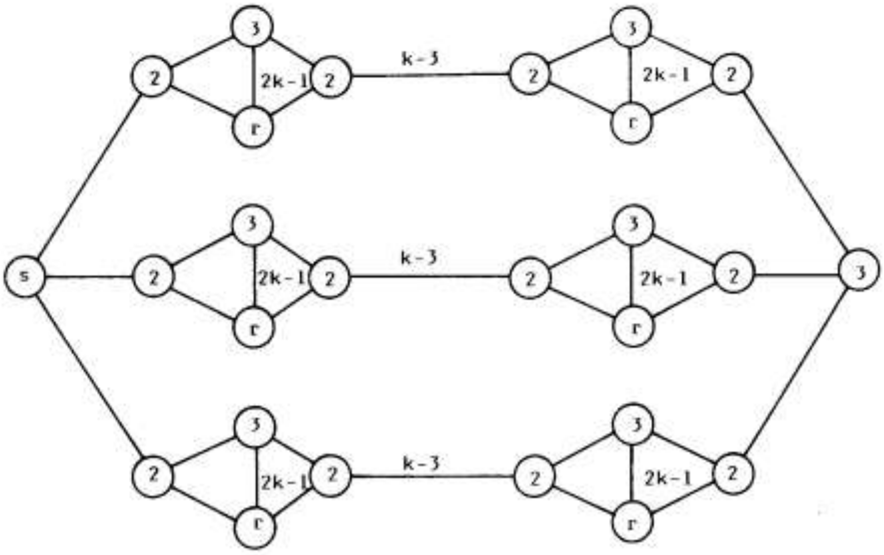

Fig. 3.5

indicates that the corresponding copies of V_m and $V_{m'}$ have a half edge identified. The success of the construction depends on the possibility of choosing the parameters m and m' so that all faces of the final map, other than interior faces of copies of V_m or D_n, are c-gons. In any case the final graph is 3-connected.

For $c = 10k+i$, $k \geqslant 1$, $i = 4, 5, 6$, a suitable multigraph is in Fig. 3.3 where $r = s = 2$ for $i = 4$, $r = 2$, $s = 3$ for $i = 5$, and $r = s = 3$ for $i = 6$. It is clear that the corresponding map M is from $S_3(5, c)$ with $s_c(\mathrm{M}) = 3$.

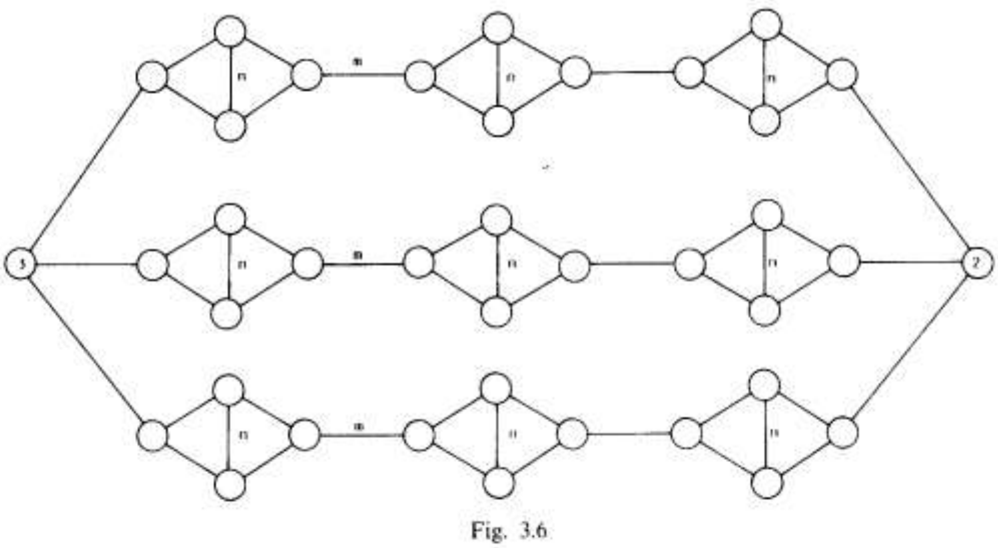

Fig. 3.6

For $c = 10k+i$, $k \geq 1$, $i = 7, 8$, a suitable map is in Fig. 3.4 where $r = 2$ for $i = 7$, and $r = 3$ for $i = 8$. The corresponding map $M \in G_3(5, c)$ has $s_c(M) = 9$.

For $c = 10k+i$, $k \geq 3$, $i = 2, 3$, we obtain the starting map M from $G_3(5, c)$ from the graph in Fig. 3.5 where $r = 2$, $s = 3$ for $i = 2$, and $r = 3$, $s = 2$ for $i = 3$. In both cases $s_c(M) = 15$.

For $c = 10k+1$, $k \geq 4$, and $c = 10k+9$, $k \geq 5$, the starting map M from $G_3(5, c)$ is in Fig. 3.6. For $c = 10k+1$ any unlabeled vertex has label 2, $m = k-4$ and $n = 2k-1$. For $c = 10k+9$, any unlabeled vertex has label 3, $m = k-5$ and $n = 2k$. In both cases we have $s_c(M) = 21$.

For $c = 6$ it follows from Jucovič [4] that there is a map $M \in G_3(5, 6)$ with $s_6(M) = d$ for any $d > 0$, $d \neq 1$.

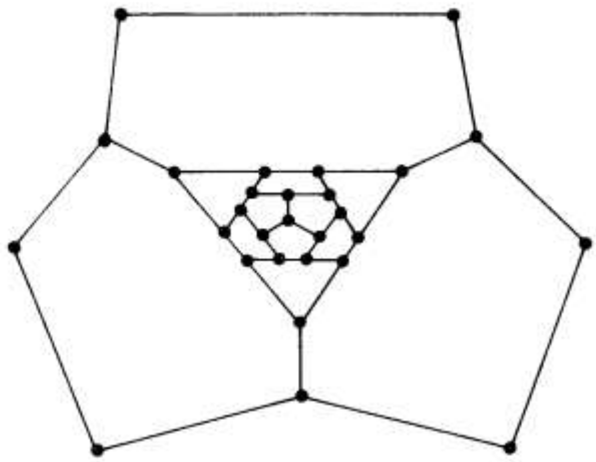

Fig. 3.7

For $c = 8$ we start with the dodecahedron. Three of its pentagons with common vertex are changed as shown in Fig. 3.7. We obtain $M \in G_3(5, 8)$ with $s_8(M) = 3$. The map M contains a quadruple of pentagons as in Fig. 3.1 (a) which can be used for performing Procedure 2.

4. The families $\mathscr{S}(3, 4, c)$ and $\mathscr{S}(3, 5, c)$

It is perhaps caused by the close connection of our procedures of construction of polytopes from $\mathscr{S}(3, 4, c)$ and from $\mathscr{S}(3, 5, c)$ that the results are so similar for these families.

Proof of the statements in Table 3.

4.1. Let us get rid of the unsuitable values v_c. The necessity of $v_c \neq 1$ follows from the nonexistence of either a 4-valent or a 5-valent planar map containing triangles and one c-gon only.

Table 3

The vertex-vectors (v_3, v_4, v_c) of polytopes from $\mathscr{S}(3, 4, c)$ and the vertex-vectors (v_3, v_5, v_c) of polytopes from $\mathscr{S}(3, 5, c)$

	c	Suitable v_c	Unsuitable v_c	Undecided v_c
1	$6k,\ k \geq 1$	all even ≥ 2	all odd	–
2	7	2, 4, 6, 8, 10, 12, all ≥ 14	1	3, 5, 7, 9, 11, 13
3	$6k+1,\ k \geq 2$	2, 4, 6, all ≥ 8	1	3, 5, 7
4	$6k+i,\ k \geq 1,\ i = 2, 3$	all ≥ 2	1	–
5	$6k+4,\ k \geq 0$	all ≥ 2	1	–
6	5	2, 4, 6, 8, 10, 12, 14, 16, 18, all ≥ 20	1	3, 5, 7, 9, 11, 13, 15, 17, 19
7	$6k+5,\ k \geq 1$	2, 4, 6, all ≥ 8	1	3, 5, 7

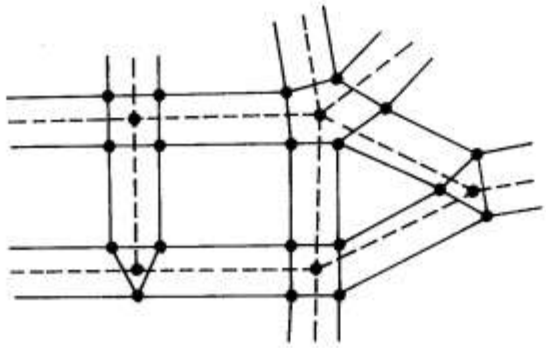

Fig. 4.1

The evenness of v_c in case $c \equiv 0 \pmod 6$ is established using Lemmas 1.1(b) and 1.2.

4.2. In constructions of polytopes proving the statements of Table 3 the following procedures for construction of 4-valent and 5-valent planar maps will be useful.

Procedure 3 is the well-known procedure *replacing edges by quadrangles.* It is shown in Fig. 4.1 where dashed lines denote the original graph M.

The obtained map M′ has the following properties: To every m-gon and every m-valent vertex in M there is associated an m-gon in M′. If two edges of M are adjacent, the corresponding quadrangles in M′ will have a common vertex. If two faces (vertices) of M are adjacent, the corresponding faces in M′ will be separated by a quadrangle. To the incident pair: an m-gon and an n-valent vertex of M, there will be associated an m-gon and an n-gon of M′ with a common vertex. Every vertex of M′ is 4-valent. For our purposes it is important that $s_i(M') = s_i(M) + v_i(M)$ for all $i \neq 4$. If in M there are triangles and c-gons and trivalent and c-valent vertices only then every edge of M′ is common to a quadrangle and a k-gon, $k = 3$, c. If the graph of M is 2-connected, the graph of M′ is 3-connected (therefore polytopal). So if M has vertices and faces of types just described, the dual of M′ belongs to $\mathscr{S}(3, 4, c)$.

Procedure 4 consists of two steps.

First by Procedure 3 the map M′ with a regular 4-valent graph is constructed.

Second step: Every quadrangle of M′ corresponding to an edge of M is divided by its diagonal into two triangles in such a way that we obtain a 5-valent map M* for which $s_i(M^*) = s_i(M) + v_i(M)$ for all $i \neq 3$. (This is always possible because of the orientability of the sphere.)

If the given planar map M′ has a 2-connected graph and contains triangles and c-gons and trivalent and c-valent vertices only, then the map M* has a regular 5-valent graph such that $s_i(M^*) = s_i(M) + v_i(M)$ for $i = 3$, c and its graph is 3-connected.

It is clear that $r(M^*)$, the radial map of M*, is from the family $\mathscr{S}(3, 5, c)$.

4.3. To prove the statements in the second column of Table 3 it is sufficient to construct suitable maps M mentioned above in Procedures 3 and 4.

Let $v_c \equiv 0 \pmod 2$. We start with the map of the tetrahedron. A con-

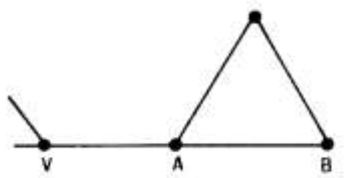

Fig. 4.2

figuration consisting of the triangle ABC and the vertex V as in Fig. 4.2 is used. The edge AC of the triangle ABC is divided by the vertices $A_1, \ldots, A_{c-3}$ into $c-2$ parts and new edges VA_i, $i = 1, \ldots, c-3$, are inserted. A pair: a c-valent vertex and a c-gon, appears. The valencies of the other vertices and faces are not changed. The obtained map again contains a pair: a triangle and a 3-valent vertex, needed for increasing the number of elements of degree c of M.

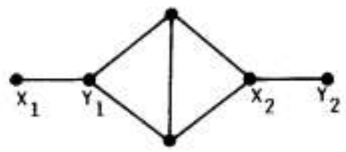

Fig. 4.3

For $v_c \equiv 1 \pmod 2$ the construction of the suitable map M depends on c by mod 6. A basic role in constructions will be played by the configuration R_k, $k \geq 1$. The configuration R_1 is shown in Fig. 4.3. The configuration R_k, $k \geq 2$, is obtained from R_{k-1} and R_1 by identifying the edge $X_2 Y_2$ of R_{k-1} with the edge $X_1 Y_1$ of R_1 and then deleting these labels. R_0 denotes an edge $X_1 Y_1$ only. Our constructions begin with 2-connected planar maps with labeled edges. An edge with label k denotes R_k. An unlabeled edge (or an edge with label 0) denotes R_0.

For $c = 6k + i$, $i = 2, 3, 4$, $c \geq 4$, $k \geq 0$, the construction starts with the map in Fig. 4.4(i). The obtained map M_0 has three c-gons. All other faces and all vertices have degree three. The further needed $v_c - 3$ elements of valency

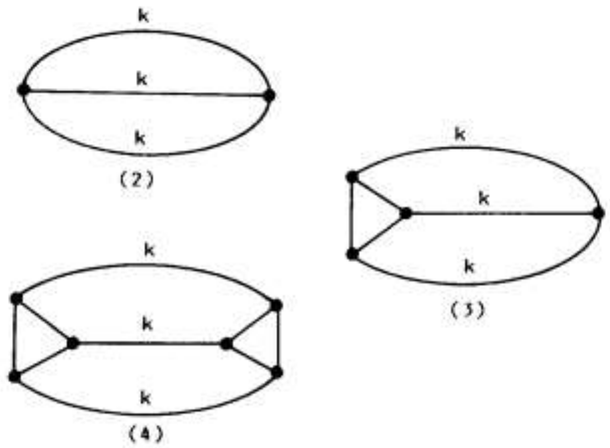

Fig. 4.4

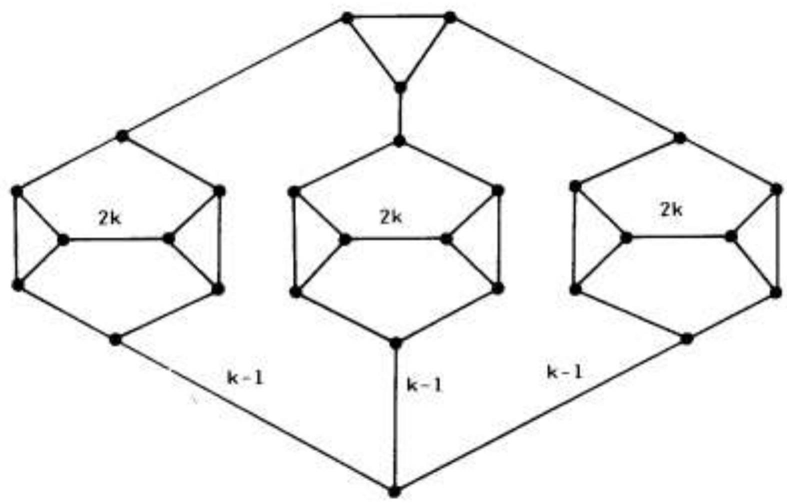

Fig. 4.5

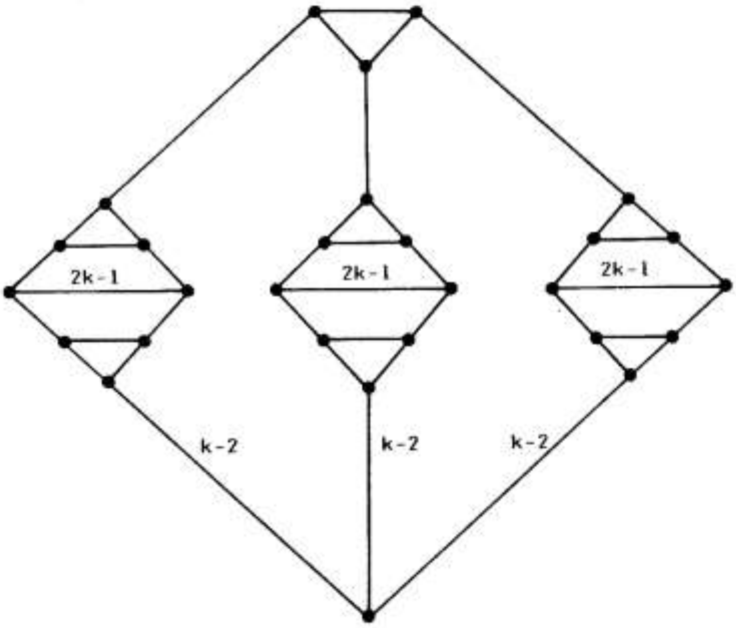

Fig. 4.6

c will be obtained by using pairs: a 3-valent vertex and a triangle, as described above.

For $c = 6k+5$, $k \geqslant 1$, the starting map M is shown in Fig. 4.5 and for $c = 6k+1$, $k \geqslant 2$, the construction begins with the map in Fig. 4.6. In both

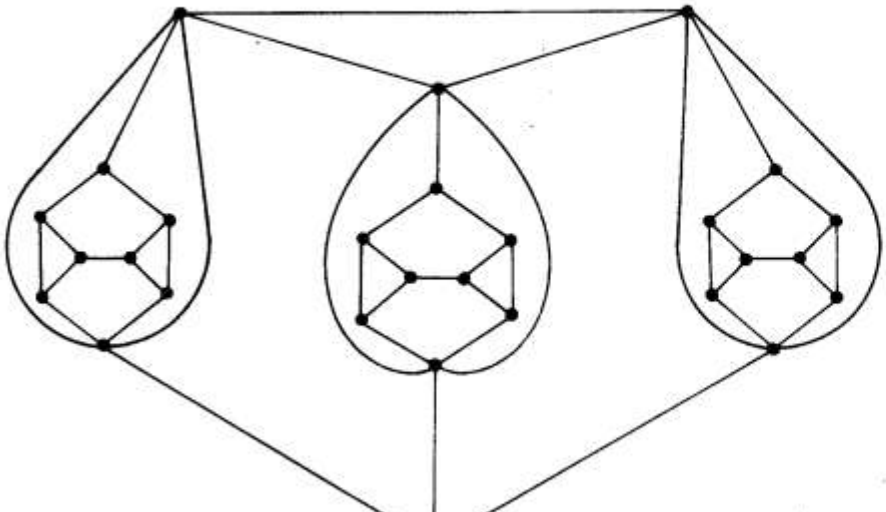

Fig. 4.7

cases the starting maps have nine c-gonal faces. The additional $v_c - 9$ elements needed will be obtained as above using pairs: a triangle and a 3-valent vertex.

For $c = 7$ the starting map with fifteen 7-gons is shown in Fig. 5.5 (consider a trivalent vertex instead of a dark marked triangle).

For $c = 5$ the starting map with 21 elements of valency five is shown in Fig. 4.7.

5. The families $\mathscr{S}(3, 3, c)$

Table 4

The vertex-vectors (v_3, v_c) of polytopes from $\mathscr{S}(3, 3, c)$

	c	Suitable v_c	Unsuitable v_c	Undecided v_c
1	4	2, 3	all $\neq 2, 3$	–
2	5	2, 6	all $\neq 2, 6$	–
3	6	all even $\geqslant 2$	all odd	–
4	7	2, all $\equiv 0 \pmod 3$ and $\geqslant 6$ and $\neq 9, 21$	3, 9, all $\not\equiv 0 \pmod 3$ and $\neq 2$	21
5	8	2, all $\equiv 0 \pmod 3$ and $\geqslant 6$ and $\neq 3, 9$	3, 9, all $\not\equiv 0 \pmod 3$ and $\neq 2$	–
6	9	2, all even and $\geqslant 12$	1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13	all odd $\geqslant 15$
7	10	2, all $\equiv 0 \pmod 3$ and $\geqslant 12$ and $\neq 15, 18, 21, 30, 33, 45$	3, 6, 9, 15, all $\not\equiv 0 \pmod 3$ and $\neq 2$	18, 21, 30, 33, 45
8	$\geqslant 11$	2	all $\neq 2$	–

Proof of the statements in Table 4.

5.1. Crucial in the proofs of unsuitability of certain values are the following lemmas.

LEMMA 5.1. *If* $\mathrm{M} \in \mathscr{S}(3, 3, c)$, *then* M *is the radial map of the c-gonal pyramid or of a planar map belonging to* $G_3(3, c)$.

Proof. No polytope $\mathrm{M} \in \mathscr{S}(3, 3, c)$ contains a quadrangle having only 3-valent vertices; otherwise M contains as a subgraph the graph of the cube which has in M at most two c-valent vertices and so the graph of M is not 3-connected, a contradiction to Steinitz's theorem concerning polyhedral graphs.

By a *3-path* (U, V) we mean a path joining two c-valent vertices U, V whose every internal vertex is 3-valent. If P is the shortest 3-path (U, V) in M, then its length is at most 3. Indeed, if this is not true and the shortest 3-path (U, V) is $U = V_0, V_1, \ldots, V_n = V$, $n \geqslant 4$, then the quadrangle $V_1 V_2 V_3 W$ has every vertex of degree 3 in contradiction to our observation at the beginning of the proof.

All vertices of M can be regularly colored by two colors. If in M there exist two c-valent vertices of different colors, then they are joined by a 3-path of of length 3. From the unambiguity of the construction it follows that M has exactly two c-valent vertices. In this case M is the radial map of the c-pyramid. If the length of the shortest 3-path is 2, then all c-valent vertices have the same color in accordance with our statement.

M is the radial map of the map with the graph G formed in the following way: The vertices of G are all vertices of M colored by colors different from those of c-valent vertices, and an edge joins two vertices if they are vertices of the same quadrangle of M. The graph G is 3-edge-connected and 2-vertex-connected because in the opposite case its radial map is not polytopal.

LEMMA 5.2. *For every* $M \in \mathscr{S}(3, 3, c)$ *with* $v_c \neq 2$ *there exists a planar map with a 3-regular graph having exactly* v_c *faces which are incident with at least* h *edges,* $h \geqslant c/2$, *each.*

We obtain the required map by replacing every triangle of the 3-valent planar map whose radial map is M by a 3-valent vertex.

From Euler's formula the following lemma follows easily:

LEMMA 5.3. *For the face-vector* $(s_3, s_4, \ldots)$ *of a planar map with a 3-regular graph we have*:

(A) *If* $s_3 \neq 0$ *and* $s_k = 0$ *for all* $k \geqslant 4$, $k \neq c$, *then* $3s_3 = 12+(c-6)s_c$.
(B) *If* $s_3 = 0$, *then* $\sum_{i \geqslant 4} s_i \geqslant 6$.
(C) *If* $s_3 = s_4 = 0$, *then* $\sum_{i \geqslant 5} s_i \geqslant 12$.
(D) *At least one face of* M *has less than* 6 *edges.*

5.2. All statements in lines 1, 2 and 3 of Table 4 follow from basic properties of planar 3-valent maps (see Grünbaum [2], Jucovič [4] and Lemma 5.3).

The unsuitability of integers $\not\equiv 0 \pmod 3$ and $\neq 2$ in lines 4, 5, 7 is a simple corollary of Lemmas 5.1, 5.2 and 5.3(A).

The unsuitability of $v_7 = 3$ and $v_8 = 3$ or $v_9 \leqslant 11$ except $v_9 = 2$ and $v_{10} = 3, 6, 9$ is a corollary of Lemmas 5.1, 5.2 and 5.3(B) or 5.3(C), respectively.

The unsuitability of integers in line 8 follows from Lemmas 5.1, 5.2 and 5.3(D).

The unsuitability of $v_7 = 9$ or $v_8 = 9$ follows from a detailed investigation of 3-valent planar maps having exactly nine 7-gons or 8-gons and triangles whose radial maps could belong to $\mathscr{S}(3, 3, c)$, which we omit here. In fact, it can be shown that they do not exist.

The proof of the unsuitability of $v_{10} = 15$ is similar.

The unsuitability of $v_9 = 13$ follows from the nonexistence of a map $M \in G_3(5, 6)$ with $s_6(M) = 1$ (cf. [4, p. 61]).

5.3. Let us prove column 2 in Table 4, the suitability of certain values. The radial polytope of the c-pyramid, $c \geqslant 4$, belongs to $\mathscr{S}(3, 3, c)$ and has the vertex-vector $(v_3 = 2c, v_c = 2)$. The suitability of any other value of v_c will be proved (using Lemma 5.1) by constructing a map $M \in G_3(3, c)$ with $s_c(M) = v_c$. The following Procedures 5 and 6 applied to certain starting maps are employed.

Procedure 5 consists in replacing a pair of triangles joined by an edge (configuration C) as in Fig. 5.1 by the cell-aggregate O_1 in Fig. 5.2 (Procedure 5a) or by the cell-aggregate O_2 in Fig. 5.3 (Procedure 5b). In both cases the map obtained contains configurations C for repeating the procedures.

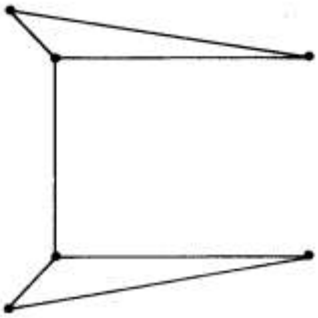
Fig. 5.1


Fig. 5.2

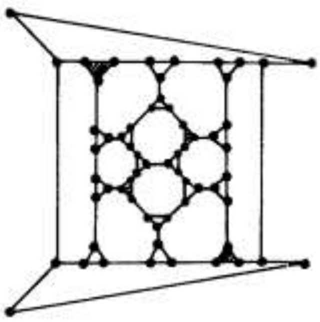
Fig. 5.3

The cell-aggregate O_1 contains six 8-gons; if the dark marked triangles are changed into trivalent vertices, then O_1 contains six 7-gons. Therefore performing once Procedure 5a causes increasing the number of 8-gons or 7-gons by six.

The cell-aggregate O_2 contains twelve 10-gons; if the dark marked triangles are changed into trivalent vertices, O_2 contains twelve 9-gons. So

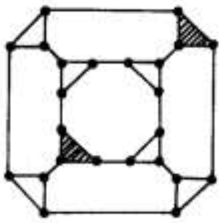

Fig. 5.4

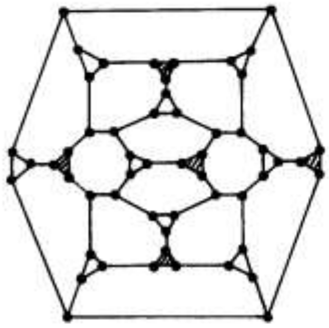

Fig. 5.5

performing once Procedure 5b causes increasing the number of 10-gons or 9-gons by twelve.

Now, performing Procedure 5a with the 8-gons or 7-gons on the map in Figs. 5.4, 5.5 or 5.6 (a) (where the dark area is the map in Fig. 5.6 (b)) proves the statements in lines 4 and 5 (for $c = 7$ the dark triangles in Figs. 5.4 and 5.5 are replaced by trivalent vertices).

Performing Procedure 5b with 9-gons on the six maps in Fig. 5.7 proves the statement in line 6 for $v_c \neq 2$.

Performing Procedure 5b with 10-gons on the map of the dodecahedron whose every vertex is replaced by a triangle proves the statement in line 7 for $v_{10} \equiv 0 \pmod{12}$. To settle the remaining statements in line 7 a new procedure is introduced.

Procedure 6 allows us to increase the number of 10-gons by 15 as follows: Having in the given map a submap as in Fig. 5.8 (configuration K) it is replaced by the cell-aggregate in Fig. 5.9; in it configuration K is contained making it possible to repeat the procedure.

Except for the number 2 every number in line 7 and column 2 can be expressed in the form $12m+15s$ ($m \geqslant 1$, $s \geqslant 0$ are integers). (The undecided

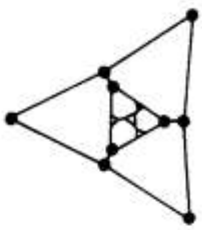

Fig. 5.6(a)

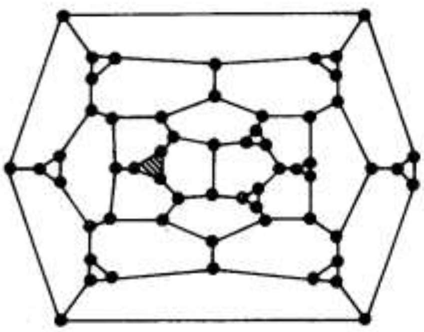

Fig. 5.6(b)

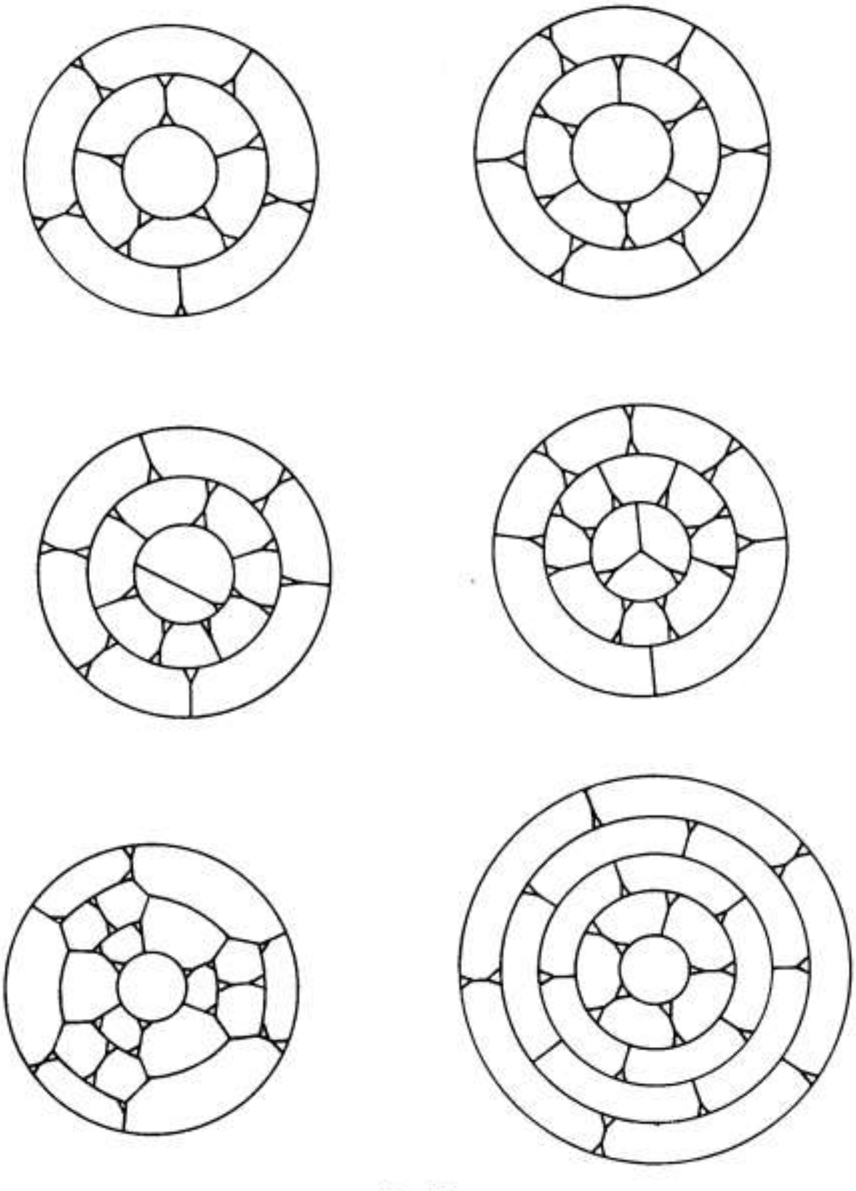

Fig. 5.7

values in that line cannot be expressed in that form.) Having the map with $12m$ 10-gons (constructed with the use of Procedure 5 which ensures the existence of a configuration K in it) we perform on it Procedure 6 s times.

The radial maps of the constructed maps belonging to $G_3(3, c)$ are the required maps, proving statements in column 2 of Table 4.

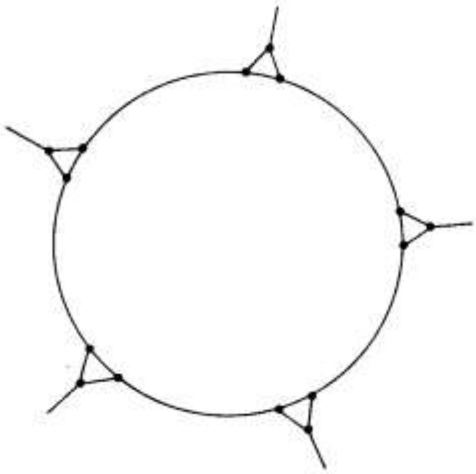

Fig. 5.8

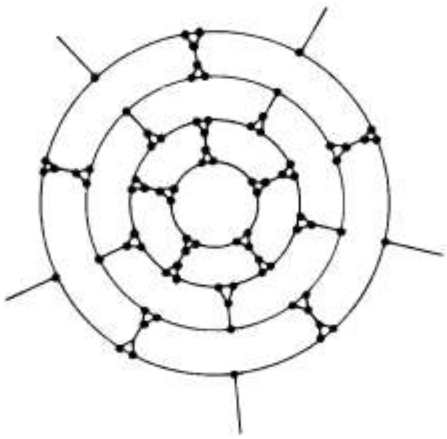

Fig. 5.9

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